Versatile, Accurate and Analytically Tractable Approximation for the Gaussian Q-function

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ABSTRACT

The existing approximations for the Gaussian Q-function have been developed bearing in mind applications that require high estimation accuracies (e.g., derivation of the error probability for digital modulation schemes). Unfortunately, the associated mathematical expressions are too complex to be easily employed in many other analytical studies, even if they do not require such a high accuracy. This letter proposes a simple, yet accurate mathematical approximation to the Gaussian Q-function. When compared to other existing approximations, the proposed model provides an adequate balance between accuracy and analytical tractability. Its simplicity enables its application over a wider range of analytical studies at reasonable accuracy levels. As an illustrative example, the proposed approximation is employed to obtain a new and simple closed-form expression for the probability of detection of an energy detector under Rayleigh fading channels.

Index terms - Gaussian Q-function; Spectrum sensing; Energy detection; Rayleigh fading.

I. INTRODUCTION

The Gaussian Q-function $\mathcal{Q}(x)$ [1, 26.2.3], the directly related error function $\mathrm{erf}(x)$ [1, 7.1.1], and its complementary error function $\mathrm{erfc}(x)$ [1, 7.1.2] are of paramount importance in many practical problems found in electrical engineering and other related fields, where unknown factors under study are frequently modeled as Gaussian random variables in order to make them mathematically tractable (the most clear example is the thermal noise present in any communication system). Unfortunately, no exact and simple closed-form expressions, appropriate for mathematical manipulations, are known for these functions. In many cases it is useful (and enough) to have closed-form approximations in order to facilitate analytical manipulations. As a result, and besides efficient numerical methods and infinite series proposed for the calculation of the Q-function [1–6], several empirical approximations have been presented in the literature [7–12] providing different trade-offs between estimation accuracy and analytical tractability. Some of them offer simple mathematical expressions that can easily be employed in analytical studies, at the cost of a limited accuracy [8]. On the other hand, some others are able to provide accurate estimates of the true Q-function's values [7]. These approximations have been developed bearing in mind applications that require high estimation accuracies (e.g., derivation of the error probability for digital modulation schemes in fading channels, which may be in the order of 10^{-6} to 10^{-9}). Unfortunately, the associated mathematical expressions are too complex to be easily employed in many other analytical studies, even if they do not require such a high accuracy. In this letter, a quite simple mathematical approximation to the Gaussian Q-function is proposed. Its simplicity enables its application over a wider range of analytical studies at a reasonable accuracy level.

II. Previous Work

Previous approximations for the Gaussian Q-function are reviewed in this section. In 1979, Börjesson et al. proposed the following tight approximation [7, eq.(13)]:

$$\mathcal{Q}(x) \approx \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(1-a)x + a\sqrt{x^2+b}} \cdot e^{-x^2/2}, \quad x \geq 0 \tag{1}$$

where $a = 0.339$ and $b = 5.510$ are computed to minimize the maximum absolute relative error.

A simpler, but less accurate approximation, was proposed by Chiani et al. [8, eq.(14)]:

$$\operatorname{erfc}(x) \approx \frac{1}{6}e^{-x^2} + \frac{1}{2}e^{-4x^2/3}, \quad x \geq 0 \tag{2}$$

Prony (sum-of-exponentials) approximations were also proposed by Loskot et al. [9, eqs.(13c/d)]:

$$\mathcal{Q}(x) \approx 0.208\, e^{-0.971\, x^2} + 0.147\, e^{-0.525\, x^2}, \qquad x \geq 0 \tag{3}$$

$$\mathcal{Q}(x) \approx 0.168\, e^{-0.876\, x^2} + 0.144\, e^{-0.525\, x^2} + 0.002\, e^{-0.603\, x^2}, \quad x \geq 0 \tag{4}$$

A novel approximation aimed at increasing the tightness in the region of small function's arguments was contributed by Karagiannidis et al. [10, eq.(6)]:

$$\operatorname{erfc}(x) \approx \frac{1-e^{-Ax}}{B} \cdot \frac{e^{-x^2}}{\sqrt{\pi}x}, \quad x \geq 0 \tag{5}$$

where A and B are computed so as to minimize the integral of the absolute error (for $x \in [0, 20]$, $A = 1.98$ and $B = 1.135$). A modified version of equation 5 was proposed by Isukapalli et al. with the aim of providing an easily integrable expression for any m of a Nakagami-m fading distribution while preserving the tightness of the approximation [11, eq.(3)]:

$$\mathcal{Q}(x) \approx e^{-x^2/2} \sum_{n=1}^{n_a} \frac{(-1)^{n+1}(A)^n}{B\sqrt{\pi}(\sqrt{2})^{n+1} n!} x^{n-1}, \quad x \geq 0 \tag{6}$$

where n_a is the number of terms considered after truncating the infinite series.

A polynomial approximation, based on the observation that a Gaussian random variable can be represented by a sum of n uniform random variables, was proposed by Chen et al. [12, eq.(4)] for analytical derivations of error rates in log-normal channels:

$$\mathcal{Q}(x) \approx 1 - \sum_{m=0}^{n} \sum_{p=0}^{n} \frac{(-1)^{m+p}\binom{n}{p}}{m!(n-m)!} \left(\frac{n}{12}\right)^{p/2} \left(\frac{n}{2} - m\right)^{n-p} x^p \, U\left(x - \sqrt{\frac{12}{n}}\left(\frac{n}{2} - m\right)\right), |x| < \sqrt{3n} \tag{7}$$

where $U(\cdot)$ represents the unit step function [1, 29.1.3].

III. Proposed Approximation

The existing approximations are well suited for their particular applications. However, they are too complex to be easily employed over a wide variety of analytical studies. To cope with this drawback, a mathematical model based on a second-order exponential function is here proposed:

$$\mathcal{Q}(x) \approx e^{ax^2+bx+c}, \quad x \geq 0 \tag{8}$$

where $a, b, c \in \mathbb{R}$ are fitting parameters. The main attractiveness of equation 8 is its analytical simplicity. Notice, for instance, that the extension to any power of $\mathcal{Q}(x)$ is trivial. However, the interest of this approximation lies not only on its mathematical simplicity but also, as it will be shown, in its ability to capture the behavior of the Q-function with significant accuracy levels.

The proposed model is characterized by only three fitting parameters, the optimum value of which can be determined according to different criteria. Two different fitting criteria are considered in this letter to this end. The first one computes the optimum values of fitting parameters (a^*, b^*, c^*)

so as to minimize the Sum of Square Errors (SSE):

$$(a^*, b^*, c^*) = \arg \min_{(a,b,c)} \left\{ \sum_{n=1}^{N} \left[\mathcal{Q}(x_n) - e^{a x_n^2 + b x_n + c} \right]^2 \right\} \tag{9}$$

where N is the number of numerical values x_n employed for the argument x in the fitting process. This approach, which minimizes the overall absolute error over the range of considered arguments, will henceforth be referred to as the *min-SSE criterion*.

The second fitting criterion consists in minimizing the Maximum Absolute Relative Error (MARE). The optimum values of fitting parameters (a^*, b^*, c^*) are therefore computed as:

$$(a^*, b^*, c^*) = \arg \min_{(a,b,c)} \left\{ \max_{n} \left\{ \left| \frac{\mathcal{Q}(x_n) - e^{a x_n^2 + b x_n + c}}{\mathcal{Q}(x_n)} \right| \right\} \right\} \tag{10}$$

This approach, which minimizes the MARE observed for all n, will henceforth be referred to as the *min-MARE criterion*.

IV. Accuracy Analysis and Comparison

The optimum values of fitting parameters (a^*, b^*, c^*) are shown in Table 1. The fitting process has been performed over different argument ranges, denoted as $\hat{x}$. These argument ranges represent the interval for which the obtained fit is optimum according to the corresponding criterion. The Goodness-Of-Fit (GOF) is evaluated in Figure 1 in terms of the absolute relative error as a function of the argument x. As it can be appreciated, the GOF for the min-MARE criterion is clearly dependent on the argument range $\hat{x}$ for which the fit is optimized, while in the case of the min-SSE criterion it is significantly independent, excepting the case $\hat{x} \in [0, 2]$ where a slightly different result is observed. The accuracy attained with min-SSE is noticeably good for small arguments ($x \leq 1.5$), but degrades as the argument values increase. On the other hand, the min-MARE criterion in general provides coarser, yet reasonably accurate estimates over a wider range of arguments. It is interesting to note, in the latter case, how the accuracy degrades for argument values above the optimized interval $\hat{x}$. Moreover, as the length of the optimized interval $\hat{x}$ decreases, the overall relative error within $\hat{x}$, and also the accuracy, improve.

The proposed approximation is compared with other existing solutions in Figures 2, 3 and 4 within an optimized interval $\hat{x} = [0, 6]$. The impact of larger optimized intervals is discussed where appropriate. Among the previously proposed approximations, the Börjesson's approximation can arguably be considered as the most accurate one for almost all arguments. Other proposed approximations, in general, are not able to provide its level of accuracy. Nevertheless, it is interesting to note, for small arguments ($x \leq 1.5$), that the min-SSE criterion provides similar accuracy levels and within a limited region ($x \leq 0.8$) even outperforms the Börjesson's approximation. From the comparison of equations 1 and 8 it is clear that the proposed exponential approximation provides a much simpler and analytically tractable expression at a reasonable accuracy level.

The Chen's polynomial approximation was proposed for analytical derivations in log-normal channels, for which exponential and rational approximations are not well suited. Besides this particular scenario, the simpler proposed approximation appears to be applicable over a wider range of analytical problems. In terms of accuracy, Figure 2 shows that the proposed approximation with the min-SSE criterion is comparable ($x \gtrsim 1.75$) or even better ($x \lesssim 1.75$). The accuracy of the Chen's approximation can be improved by increasing n in equation 7. However, an important increase of n does not seem to provide a significant accuracy improvement. Therefore, the proposed approximation is able to provide a comparable (even better) accuracy level with a much simpler mathematical expression (see equations 7 and 8).

When compared to the Karagiannidis' approximation, the obtained accuracy depends on the considered configuration and argument range. The min-SSE criterion provides better accuracy for low arguments ($x \leq 1.88$) while the min-MARE criterion provides better accuracy for large arguments ($x \geq 2.14$ for $\hat{x} \in [0, 6]$, $x \geq 3.21$ for $\hat{x} \in [0, 10]$, $x \geq 6.33$ for $\hat{x} \in [0, 20]$). Between both argument ranges, there exists a limited region where the Karagiannidis' approximation provides a slightly better accuracy, which is comparable to that of the proposed exponential approximation. Depending on the particular argument range involved in the problem under study, the proposed exponential approximation can be configured to provide a comparable level of accuracy with a simpler and more tractable analytical expression (see equations 5 and 8).

The Isukapalli's approximation constitutes a simplification of the Karagiannidis' approximation, by means of a truncated infinite series, with the explicit purpose of analytical tractability. Hence, it is expected that the former cannot outperform the latter in terms of accuracy, which is corroborated in Figure 3. In fact, it can be verified, as expected, that the accuracy of equation 6 tends asymptotically to that of equation 5 as $n_a \to \infty$. As a result, accuracy improvements observed for the proposed approximation with respect to the Karagiannidis' approximation are also guaranteed with respect to the Isukapalli's approximation. In terms of analytical complexity, the proposed approximation provides a mathematical expression simpler than the Isukapalli's approximation, thus resulting in a more adequate balance between accuracy and analytical tractability.

Finally, the main attractiveness of the Chiani's approximation is its analytical simplicity, which comes at the cost of limited accuracy (see Figure 4). The obtained results indicate that the proposed exponential approximation with the min-MARE criterion provides a mathematical expression with a similar complexity, in some cases even slightly simpler (only one single exponential term), that clearly outperforms the Chiani's approximation for almost all arguments. The Loskot's approximation with two exponential terms outperforms the Chiani's approximation with the same analytical complexity. However, the proposed approximation still provides a more adequate balance between accuracy and analytical tractability since it attains a comparable level of accuracy with a single exponential term. Although the accuracy of the Loskot's approximation can be enhanced with a third term, the resulting improvement comes at the expense of a slightly increased complexity.

Based on the previous analysis, it can be concluded that the proposed exponential approximation for the Gaussian Q-function constitutes an adequate balance between accuracy and analytical tractability. On one hand, such approximation provides a simpler analytical expression than other proposals, thus enabling its use over a wider range of analytical studies. On the other hand, the resulting accuracy is similar or even better than that attained by other existing proposals based on more complex mathematical expressions.

V. APPLICABILITY EXAMPLE

Previous approximations to the Gaussian Q-function have mainly been developed to evaluate the bit, symbol or block error probabilities in many communication theory problems. The approximation proposed in this letter could be employed to solve many of such problems as well. However, in order to illustrate its versatility and applicability, this section considers a different problem related to the field of Dynamic Spectrum Access (DSA) in Cognitive Radio (CR) networks.

The DSA/CR paradigm has been identified as a promising solution to solve the existing conflicts between spectrum demand growth and spectrum underutilization. The basic underlying idea of DSA/CR is to allow unlicensed users to access in an opportunistic and non-interfering manner some licensed bands temporarily unoccupied by licensed users. To guarantee interference-free spectrum access, DSA/CR networks must be able to reliably identify the presence of licensed users. To this end, various signal detection methods, referred to as spectrum sensing techniques, have been considered. One of the most simple and widely employed spectrum sensing algorithms is Energy Detection (ED). This section employs the approximation proposed in this letter to easily compute the ED's average probability of detection under Rayleigh fading environments. This problem was already studied in [13], resulting in mathematical expressions of notable complexity. This section obtains a much simpler alternative closed-form expression and illustrates the potential applicability and usefulness of the proposed approximation with a practical case study.

The probability of detection of ED can be approximated as [14]:

$$P_d(\gamma) = \mathcal{Q}\left(\frac{\mathcal{Q}^{-1}(P_{fa})\sqrt{2N} - N\gamma}{\sqrt{2N(1+\gamma)^2}}\right) = \mathcal{Q}\left(\zeta(\gamma)\right) \tag{11}$$

where N is the number of signal samples collected during the sensing interval, P_{fa} is the target probability of false alarm, γ is the Signal-to-Noise Ratio (SNR) and, assuming the common case of low SNR regime in DSA/CR ($\gamma \ll 1$), $\zeta(\gamma)$ can be approximated as:

$$\zeta(\gamma) = \frac{\mathcal{Q}^{-1}(P_{fa})\sqrt{2N} - N\gamma}{\sqrt{2N(1+\gamma)^2}} \approx \mathcal{Q}^{-1}(P_{fa}) - \sqrt{\frac{N}{2}}\gamma \tag{12}$$

Equation 11 gives the probability of detection conditioned on the instantaneous value of γ. Under varying SNR, a more useful performance parameter is the average probability of detection $\overline{P}_d$ experienced for an average SNR γ_0, which may be derived by averaging equation 11 over the SNR statistics:

$$\overline{P}_d(\gamma_0) = \mathbb{E}\left[P_d(\gamma)\right] = \int_\gamma P_d(\gamma) f_\gamma(\gamma) d\gamma = \int_\gamma \mathcal{Q}\left(\zeta(\gamma)\right) f_\gamma(\gamma) d\gamma \tag{13}$$

where $f_\gamma(\gamma)$ is the Probability Density Function (PDF) of the received SNR. If the signal amplitude follows a Rayleigh distribution as usual in mobile communication systems, the SNR then follows an exponential PDF given by:

$$f_{\gamma}(\gamma) = \frac{1}{\gamma_0} \exp\left(-\frac{\gamma}{\gamma_0}\right), \quad \gamma \geq 0 \tag{14}$$

Equation 13 can be simplified and solved by applying the approximation suggested in equation 8. Notice that $\zeta(\gamma)$, the argument of $\mathcal{Q}(x)$, may take both positive and negative values even though $\gamma \geq 0$ (see equation 12). Since the approximation in equation 8 is valid for positive arguments only, the property $\mathcal{Q}(x) = 1 - \mathcal{Q}(-x)$ must therefore be used for negative values of $\zeta(\gamma)$. Thus:

$$\begin{aligned}
\overline{P}_d(\gamma_0) &= \int_{\zeta(\gamma)<0} \left[1 - \mathcal{Q}\left(-\zeta(\gamma)\right)\right] f_{\gamma}(\gamma) d\gamma + \int_{\zeta(\gamma)\geq 0} \mathcal{Q}\left(\zeta(\gamma)\right) f_{\gamma}(\gamma) d\gamma \\
&\approx \int_{\sqrt{\frac{2}{N}}\mathcal{Q}^{-1}(P_{fa})}^{\infty} \frac{1}{\gamma_0} \exp\left(-\frac{\gamma}{\gamma_0}\right) d\gamma \\
&\quad - \int_{\sqrt{\frac{2}{N}}\mathcal{Q}^{-1}(P_{fa})}^{\infty} \frac{1}{\gamma_0} \exp\left(-\frac{\gamma}{\gamma_0}\right) \exp\left(a[\zeta(\gamma)]^2 - b\,\zeta(\gamma) + c\right) d\gamma \\
&\quad + \int_{0}^{\sqrt{\frac{2}{N}}\mathcal{Q}^{-1}(P_{fa})} \frac{1}{\gamma_0} \exp\left(-\frac{\gamma}{\gamma_0}\right) \exp\left(a[\zeta(\gamma)]^2 + b\,\zeta(\gamma) + c\right) d\gamma \\
&\approx \int_{\sqrt{\frac{2}{N}}\mathcal{Q}^{-1}(P_{fa})}^{\infty} \frac{1}{\gamma_0} \exp\left(-\frac{\gamma}{\gamma_0}\right) d\gamma \\
&\quad - \int_{\sqrt{\frac{2}{N}}\mathcal{Q}^{-1}(P_{fa})}^{\infty} \frac{1}{\gamma_0} \exp\left(-\Omega\gamma^2 - \Xi\gamma - \Theta\right) d\gamma \\
&\quad + \int_{0}^{\sqrt{\frac{2}{N}}\mathcal{Q}^{-1}(P_{fa})} \frac{1}{\gamma_0} \exp\left(-\Omega\gamma^2 - \Psi\gamma - \Phi\right) d\gamma
\end{aligned} \tag{15}$$

where $\Omega = -aN/2$, $\Xi = a\mathcal{Q}^{-1}(P_{fa})\sqrt{2N} - b\sqrt{N/2} + 1/\gamma_0$, $\Psi = a\mathcal{Q}^{-1}(P_{fa})\sqrt{2N} + b\sqrt{N/2} + 1/\gamma_0$, $\Theta = -a[\mathcal{Q}^{-1}(P_{fa})]^2 + b\mathcal{Q}^{-1}(P_{fa}) - c$, and $\Phi = -a[\mathcal{Q}^{-1}(P_{fa})]^2 - b\mathcal{Q}^{-1}(P_{fa}) - c$. Taking into account the approximation in equation 12, equation 15 is composed of three definite exponential integrals that can readily be solved in terms of exponential, error and complementary error functions:

$$\begin{aligned}
\overline{P}_d(\gamma_0) &\approx \exp\left(-\frac{1}{\gamma_0}\sqrt{\frac{2}{N}}\mathcal{Q}^{-1}(P_{fa})\right) \\
&\quad - \frac{1}{2\gamma_0}\sqrt{\frac{\pi}{\Omega}} \exp\left(\frac{\Xi^2}{4\Omega} - \Theta\right) \operatorname{erfc}\left(\sqrt{\frac{2\Omega}{N}}\mathcal{Q}^{-1}(P_{fa}) + \frac{\Xi}{2\sqrt{\Omega}}\right) \\
&\quad + \frac{1}{2\gamma_0}\sqrt{\frac{\pi}{\Omega}} \exp\left(\frac{\Psi^2}{4\Omega} - \Phi\right) \left[\operatorname{erf}\left(\sqrt{\frac{2\Omega}{N}}\mathcal{Q}^{-1}(P_{fa}) + \frac{\Psi}{2\sqrt{\Omega}}\right) - \operatorname{erf}\left(\frac{\Psi}{2\sqrt{\Omega}}\right)\right]
\end{aligned} \tag{16}$$

Although the analytical result of equation 16 is slightly complex for mathematical manipulations, it is important to note that solving equation 13 with some of the approximations presented in Section II. would have lead to much more complex solving procedures and resulting mathematical expressions. This indicates the ability of the approximation proposed in equation 8 to simplify the analytical resolution of some problems for which the approximations of Section II. were not envisaged. It is worth noting, moreover, that the two last terms of equation 16 lead to similar numerical values, specially for high values of SNR, so that they approximately cancel out each other. As a result, the simplification given in equation 17, which is valid over a wide range of SNR values (and tighter for high N and low P_{fa} values), can be employed in analytical studies:

$$\overline{P}_d(\gamma_0) \approx \exp\left(-\frac{1}{\gamma_0}\sqrt{\frac{2}{N}}\mathcal{Q}^{-1}(P_{fa})\right) \tag{17}$$

To comparatively assess the accuracy of the proposed approximation in evaluating the performance of ED under Rayleigh fading, equation 13 was solved numerically employing the exact values of the Gaussian Q-function as well as the approximated values provided by equations 1–8. The results are shown in Table 2. As it can be appreciated, all the analyzed approximations provide exact values for high SNR values but diverge as the SNR value decreases. This can be explained by the fact that $\zeta(\gamma)$, the argument of the Q-function in equation 13, increases as the SNR decreases and some of the considered approximations provide higher estimation errors for larger arguments (see Figures 2, 3 and 4). Therefore, in the case study of this section, the accuracy of the analyzed approximations is better appreciated at low SNR values. In such region, it can be observed that the Börjesson and Chiani approximations result, respectively, in the most and least accurate estimates, which is in accordance with the relative errors observed in Figures 2 and 4. Regarding the proposed approximation, the best estimate is provided in this case by the min-SSE criterion. Although other approximations are able to provide more accurate results, the accuracy of the proposed approximation can arguably be considered as sufficient for most practical problems. Moreover, it is worth emphasizing that the analytical evaluation of equation 13 with other approximations would have resulted in much more complex solving procedures, which highlights the ability of the proposed approximation to simplify analytical studies at reasonable accuracy levels.

VI. CONCLUSION

This letter has proposed a novel exponential approximation for the Gaussian Q-function providing an adequate balance between accuracy and analytical tractability. The approximation provides a simple mathematical expression that enables its use over a wide range of analytical studies. The resulting accuracy is similar or even better than that attained by other existing proposals based on more complex mathematical expressions. As an example of its versatility, the proposed approximation has been employed to contribute a new closed-form approximation to the probability of detection of an energy detector under Rayleigh fading environments.

ACKNOWLEDGEMENTS

This work was supported by the Spanish Ministry of Science and Innovation under FPU grant AP2006-848.

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		$\hat{x} \in [0, 2]$	$\hat{x} \in [0, 4]$	$\hat{x} \in [0, 6]$	$\hat{x} \in [0, 8]$	$\hat{x} \in [0, 10]$	$\hat{x} \in [0, 20]$
min-SSE	a^*	-0.3807	-0.3847	-0.3846	-0.3846	-0.3845	-0.3842
	b^*	-0.7674	-0.7632	-0.7633	-0.7634	-0.7635	-0.7640
	c^*	-0.6960	-0.6966	-0.6966	-0.6966	-0.6966	-0.6964
min-MARE	a^*	-0.3976	-0.4369	-0.4577	-0.4698	-0.4774	-0.4920
	b^*	-0.7418	-0.6511	-0.5695	-0.5026	-0.4484	-0.2887
	c^*	-0.7019	-0.7358	-0.7864	-0.8444	-0.9049	-1.1893

Table 1: Optimum values of fitting parameters (a^*, b^*, c^*) for different fitting criteria and optimized argument ranges $\hat{x}$.

		SNR (dB)						
		-20	-15	-10	-5	0	5	10
Approximation	Exact	0.0205	0.0929	0.3886	0.7263	0.9019	0.9677	0.9896
	min-SSE	0.0210	0.0931	0.3887	0.7263	0.9019	0.9677	0.9896
	min-MARE	0.0219	0.0935	0.3884	0.7263	0.9019	0.9677	0.9896
	Börjesson	0.0206	0.0929	0.3886	0.7263	0.9019	0.9677	0.9896
	Chiani	0.0255	0.0998	0.3916	0.7269	0.9020	0.9677	0.9896
	Loskot (2 terms)	0.0203	0.0927	0.3883	0.7263	0.9019	0.9677	0.9896
	Loskot (3 terms)	0.0206	0.0906	0.3865	0.7259	0.9019	0.9677	0.9896
	Karagiannidis	0.0202	0.0930	0.3888	0.7264	0.9019	0.9677	0.9896
	Isukapalli ($n_a = 8$)	0.0196	0.0926	0.3888	0.7265	0.9020	0.9677	0.9896
	Isukapalli ($n_a = 20$)	0.0202	0.0930	0.3888	0.7264	0.9019	0.9677	0.9896
	Chen ($n = 8$)	0.0198	0.0931	0.3890	0.7264	0.9019	0.9677	0.9896
	Chen ($n = 20$)	0.0203	0.0930	0.3788	0.7264	0.9019	0.9677	0.9896

Table 2: Comparison of the exact and approximated probability of detection of an energy detector under Rayleigh fading ($P_{fa} = 0.01$, $N = 1000$).

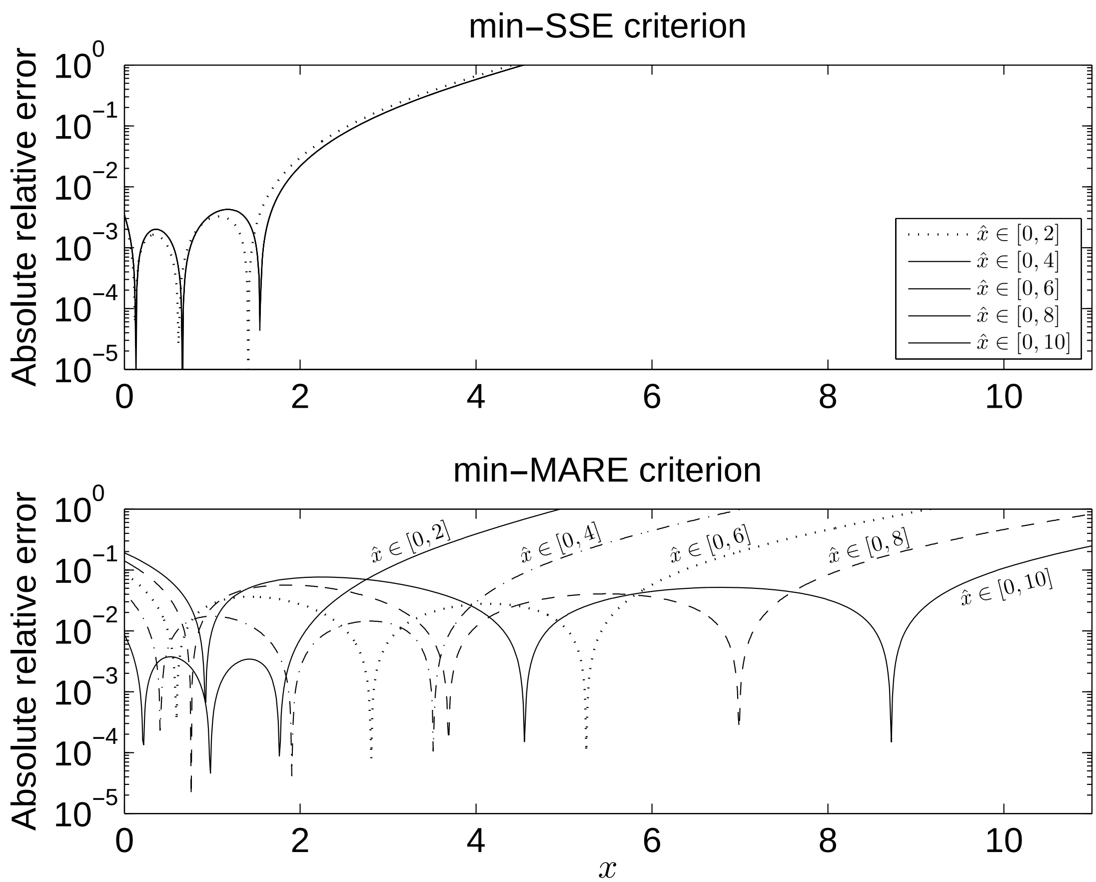

Figure 1: Absolute relative error of the proposed exponential approximations for various optimized argument ranges $\hat{x}$.

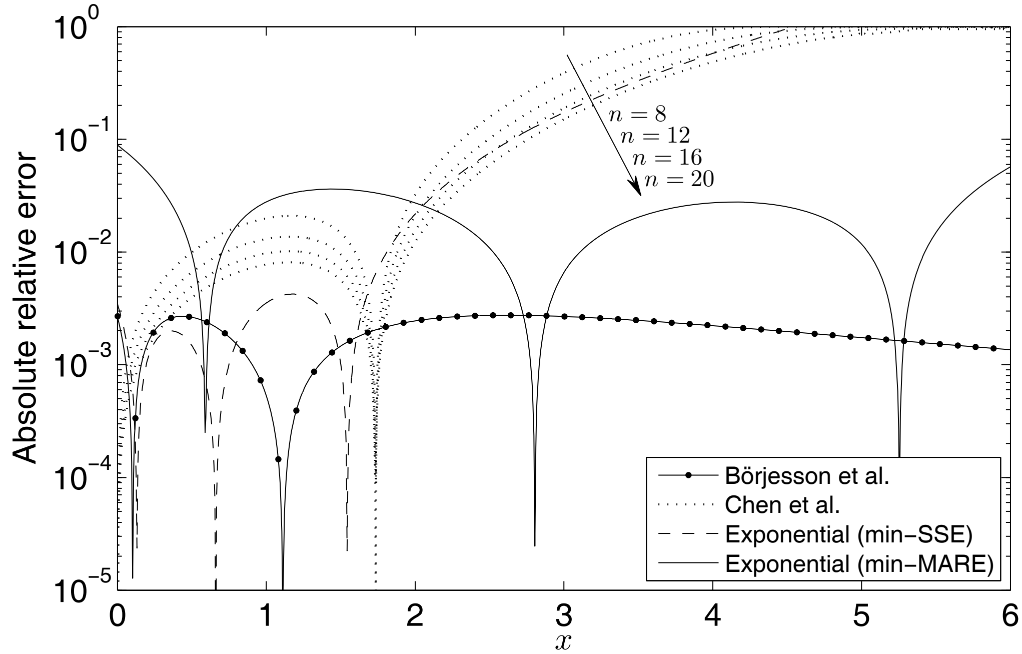

Figure 2: Absolute relative error of the proposed exponential, Börjesson et al., and Chen et al. approximations ($\hat{x} \in [0, 6]$).

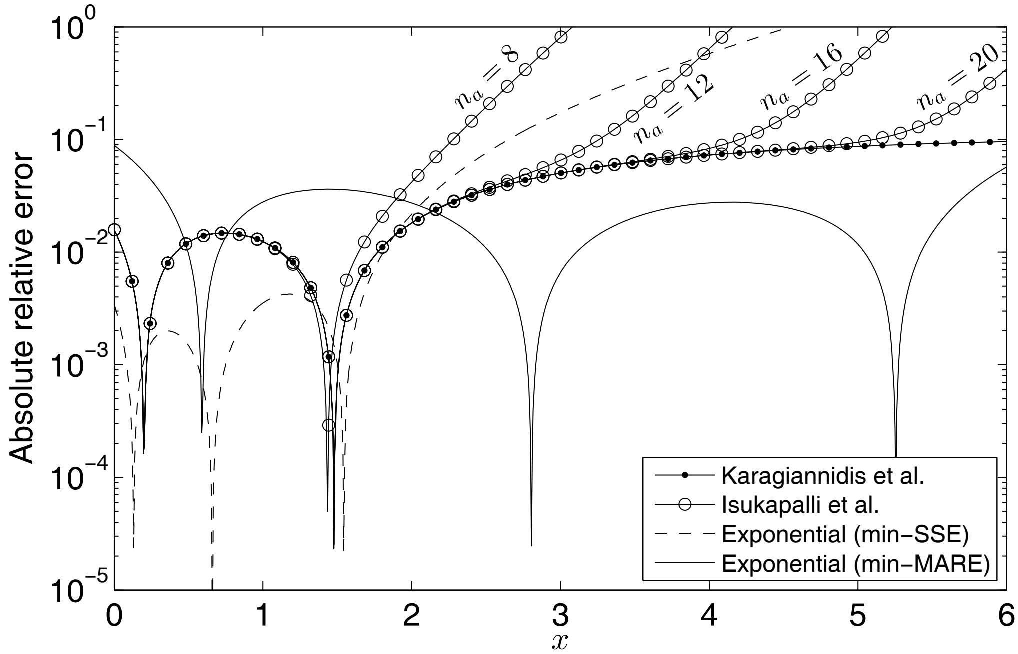

Figure 3: Absolute relative error of the proposed exponential, Karagiannidis et al., and Isukapalli et al. approximations ($\hat{x} \in [0, 6]$).

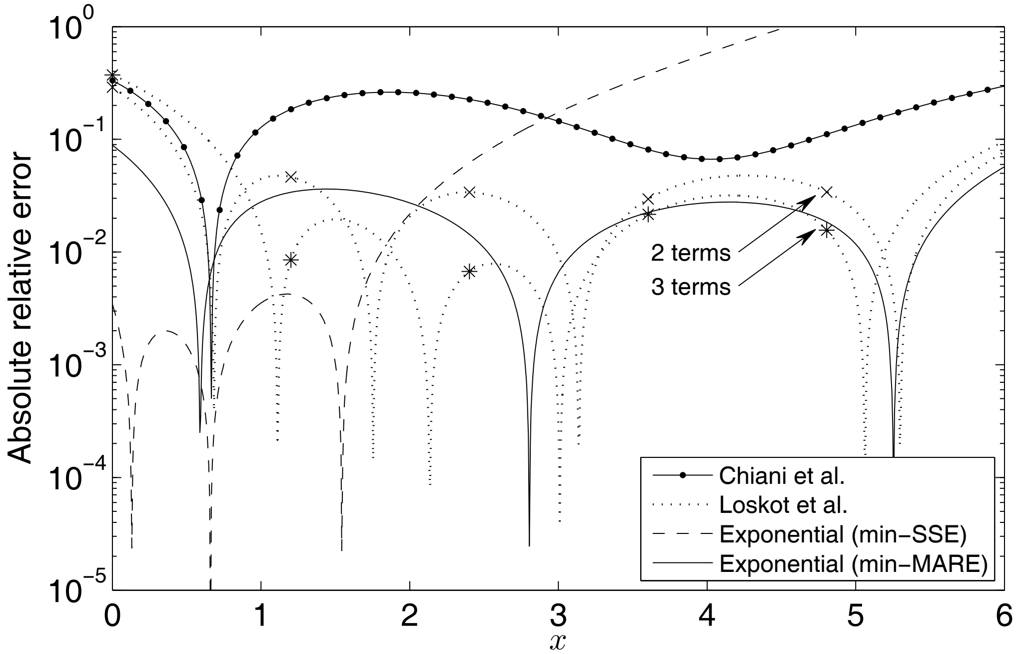

Figure 4: Absolute relative error of the proposed exponential, Chiani et al., and Loskot et al. approximations ($\hat{x} \in [0, 6]$).