Average of arbitrary powers of the Gaussian Q-function over η-μ and κ-μ fading channels

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The average of integer powers of the Gaussian Q-function over fading distributions is commonly found in the evaluation of the average error performance of wireless communication systems over fading channels. Expressions known in the literature for η-μ and κ-μ fading channels only include solutions for the average of one Gaussian Q-function and the product of two Gaussian Q-functions. This Letter overcomes this limitation by providing a general solution for the average of arbitrary powers of the Gaussian Q-function over η-μ and κ-μ fading channels.

Introduction: The η-μ and κ-μ distributions [1] have become popular models for small-scale fading in wireless communication channels under non-line-of-sight and line-of-sight conditions, respectively. The relevance and main attractiveness of these distributions relies on their complete characterisation in terms of measurable physical fading parameters and their ability to accurately fit experimental data. As a matter of fact, other popular fading models such as one-sided Gaussian, Rayleigh, Nakagami-m, Nakagami-q (Hoyt) and Nakagami-n (Rice) [2] can be obtained as particular cases of these distributions. The analysis and performance evaluation of wireless communication systems under these fading models have been the subject of increasing research interest over the last years.

An important metric extensively used in the performance evaluation of digital communication systems over fading channels is the average bit or symbol error probability, which is closely related with the particular combination of modulation format and detection method. In a broad variety of scenarios, the conditional bit and symbol error probabilities are given by expressions in terms of the Gaussian Q-function [2]. Therefore, the corresponding average error probability under fading conditions can be obtained by averaging (i.e., integrating) the Gaussian Q-function over the fading statistics. In most cases, in general (and in the case of η-μ and κ-μ fading in particular) the direct evaluation of the resulting integral [2, eq. (5.1)] is difficult owing to its algebraic complexity. As a result, an alternative and equivalent method based on the integration of the moment generating function (MGF) [2, eq. (5.3)] has commonly been used in the literature. Concretely, closed-form expressions for the MGF of the η-μ and κ-μ fading distributions have been derived in [3] and used to evaluate the average of the Gaussian Q-function under these fading channels. A simpler form for the MGF has been obtained in [4] and employed in [5] to evaluate the average not only of the Gaussian Q-function but also of the product of two Gaussian Q-functions (a form that typically appears in certain scenarios). These results are certainly useful in the average error performance evaluation of wireless communication systems under fading channels. However, in certain scenarios the conditional bit and symbol error probabilities are given in terms of integer powers of the Gaussian Q-function (e.g., see [2, eq. (8.39)]). To the best of the author's knowledge, no closed-form expressions for this scenario are known in the literature.

In this context, this Letter derives (by direct integration over the fading statistics) closed-form expressions for the average of arbitrary powers of the Gaussian Q-function over η-μ and κ-μ fading channels. The obtained results embrace the average of one and the product of two Gaussian Q-functions as particular cases. The comparison with simulation results demonstrates the accuracy of the obtained expressions and therefore their utility in the average error performance evaluation of wireless communication systems under the considered fading models.

The η-μ and κ-μ fading models: Under η-μ fading the instantaneous signal-to-noise ratio (SNR) per symbol, γ, is distributed as [1, eq. (26)]:

$$f_\gamma^{\eta\mu}(\gamma) = \frac{2\sqrt{\pi}\mu^{\mu+\frac{1}{2}}h^\mu}{\Gamma(\mu)H^{\mu-\frac{1}{2}}}\frac{\gamma^{\mu-\frac{1}{2}}}{\overline{\gamma}^{\mu+\frac{1}{2}}}\exp\left(-2\mu h\frac{\gamma}{\overline{\gamma}}\right)I_{\mu-\frac{1}{2}}\left(2\mu H\frac{\gamma}{\overline{\gamma}}\right) \quad (1)$$

where $\overline{\gamma}$ is the average SNR per symbol, η and μ are the fading parameters (h and H depend on η), $\Gamma(\cdot)$ is the gamma function [6, eq. (8.310.1)], and $I_\nu(\cdot)$ is the νth-order modified Bessel function of the first kind [6, eq. (8.431)]. The distribution is presented in two formats. In Format 1, $h=(2+\eta^{-1}+\eta)/4$ and $H=(\eta^{-1}-\eta)/4$, where $\eta>0$ denotes the ratio between the powers of the in-phase and quadrature scattered waves in each multipath cluster. In Format 2, $h=1/(1-\eta^2)$ and $H=\eta/(1-\eta^2)$, where $-1<\eta<1$ denotes the correlation between the powers of the in-phase and quadrature scattered waves in each multipath cluster. In both formats, $\mu>0$ denotes half the number of multipath clusters. This model comprises as particular cases the Nakagami-q/Hoyt ($\mu=0.5$) and Nakagami-m ($\eta\to 0$, $\eta\to\infty$, $\eta\pm 1$) distributions.

Under κ-μ fading the instantaneous SNR per symbol, γ, is distributed according to [1, eq. (10)]:

$$\begin{aligned} f_\gamma^{\kappa\mu}(\gamma) &= \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}}\exp(\mu\kappa)}\frac{\gamma^{\frac{\mu-1}{2}}}{\overline{\gamma}^{\frac{\mu+1}{2}}} \\ &\quad\times\exp\left(-\mu(1+\kappa)\frac{\gamma}{\overline{\gamma}}\right)I_{\mu-1}\left(2\mu\sqrt{\kappa(1+\kappa)\frac{\gamma}{\overline{\gamma}}}\right) \end{aligned} \quad (2)$$

where $\kappa>0$ represents the ratio between the total power of the dominant components and the total power of the scattered waves and $\mu>0$ denotes the number of multipath clusters. This model comprises the Nakagami-m ($\kappa\to 0$) and Nakagami-n/Rice ($\mu=1$) distributions.

Average of arbitrary powers of the Gaussian Q-function: In its most general form, the conditional bit and symbol error probability of wireless communication systems under additive white Gaussian noise is given by expressions involving terms of the form $\mathcal{Q}^m(\alpha\sqrt{\gamma})$, where:

$$\mathcal{Q}(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt \quad (3)$$

is the Gaussian Q-function, $m\in\mathbb{N}^+$, and $\alpha>0$ is a constant that depends on the combination of modulation format and detection method [2]. The average error probability under fading can thus be obtained by evaluating:

$$\begin{aligned} \mathcal{I}_m &= \int_0^\infty \mathcal{Q}^m(\alpha\sqrt{\gamma})f_\gamma(\gamma)d\gamma \quad (4) \\ &\approx \int_0^\infty e^{-am\alpha^2z^2-bm\alpha z-cm}f_\gamma(z^2)\,2z\,dz \quad (5) \end{aligned}$$

where the more convenient form of (5) is obtained by introducing the change of variable $\gamma=z^2$ and the approximation $\mathcal{Q}(x)\approx e^{-ax^2-bx-c}$, where (a,b,c) are fitting coefficients (see [7] and Table I therein). Notice that the algebraic form resulting from the arbitrary power m complicates the evaluation of the integral in (4). However, the introduction of the approximation to the Gaussian Q-function proposed in [7], which can easily be extended to arbitrary powers, leads to the more tractable form of (5). While (5) is an approximated expression, the corresponding result provides an excellent (virtually perfect) accuracy as will be shown.

Introducing [6, eq. (8.445)] and (1) and into (5), the resulting integral can be solved with the aid of [6, eq. (3.462.1)], which yields:

$$\begin{aligned} \mathcal{I}_m^{\eta\mu} &\approx 4\sqrt{\pi}e^{\frac{(bm\alpha)^2}{8\Upsilon}-cm}\sum_{k=0}^\infty\frac{h^\mu H^{2k}}{k!}\frac{\Gamma(4\mu+4k)}{\Gamma(\mu)\Gamma(\mu+k+\frac{1}{2})} \\ &\quad\times\left(\frac{\mu}{2\Upsilon\overline{\gamma}}\right)^{2\mu+2k}D_{-4\mu-4k}\left(\frac{bm\alpha}{\sqrt{2\Upsilon}}\right) \end{aligned} \quad (6)$$

where $\Upsilon=am\alpha^2+(2\mu h)/\overline{\gamma}$ and $D_\nu(\cdot)$ denotes the parabolic cylinder function [6, eq. (9.240)]. Truncating the infinite series to its first term (i.e., $k=0$) yields the simpler form:

$$\mathcal{I}_m^{\eta\mu}\approx 4\sqrt{\pi}e^{\frac{(bm\alpha)^2}{8\Upsilon}-cm}\frac{\Gamma(4\mu)}{\Gamma(\mu)\Gamma(\mu+\frac{1}{2})}\left(\frac{\mu\sqrt{h}}{2\Upsilon\overline{\gamma}}\right)^{2\mu}D_{-4\mu}\left(\frac{bm\alpha}{\sqrt{2\Upsilon}}\right) \quad (7)$$

which under high SNR conditions ($\overline{\gamma}\to\infty$) can be further simplified to the following asymptotic approximation:

$$\mathcal{I}_m^{\eta\mu}\approx 4\sqrt{\pi}e^{\frac{b^2m}{8a}-cm}\frac{\Gamma(4\mu)}{\Gamma(\mu)\Gamma(\mu+\frac{1}{2})}\left(\frac{\mu\sqrt{h}}{2am\alpha^2\overline{\gamma}}\right)^{2\mu}D_{-4\mu}\left(b\sqrt{\frac{m}{2a}}\right) \quad (8)$$

Notice that (8) is a power function of $\overline{\gamma}$, which constitutes a simple and tractable expression that can easily be manipulated in the evaluation of the error performance under high SNR conditions (i.e., the main SNR region of interest in the design of wireless communication systems).

Using (2) and following the same procedure, the following result is obtained for the κ-μ distribution:

$$\mathcal{I}_m^{\kappa\mu} \approx 2\, e^{\frac{(bm\alpha)^2}{8\Omega} - cm} \sum_{k=0}^{\infty} \frac{(\mu\kappa)^k e^{-\mu\kappa}}{k!} \frac{\Gamma(2\mu+2k)}{\Gamma(\mu+k)} \times \left(\frac{\mu(1+\kappa)}{2\Omega\overline{\gamma}} \right)^{\mu+k} D_{-2\mu-2k}\left(\frac{bm\alpha}{\sqrt{2\Omega}} \right) \quad (9)$$

where $\Omega = am\alpha^2 + (\mu(1+\kappa))/\overline{\gamma}$. Truncating the infinite series to its first term (i.e., $k=0$) yields the simpler form:

$$\mathcal{I}_m^{\kappa\mu} \approx 2\, e^{\frac{(bm\alpha)^2}{8\Omega} - cm} \frac{\Gamma(2\mu)}{\Gamma(\mu)} \left(\frac{\mu(1+\kappa)e^{-\kappa}}{2\Omega\overline{\gamma}} \right)^{\mu} D_{-2\mu}\left(\frac{bm\alpha}{\sqrt{2\Omega}} \right) \quad (10)$$

which under high SNR conditions ($\overline{\gamma} \to \infty$) simplifies to:

$$\mathcal{I}_m^{\kappa\mu} \approx 2\, e^{\frac{b^2 m}{8a} - cm} \frac{\Gamma(2\mu)}{\Gamma(\mu)} \left(\frac{\mu(1+\kappa)e^{-\kappa}}{2am\alpha^2\overline{\gamma}} \right)^{\mu} D_{-2\mu}\left(b\sqrt{\frac{m}{2a}} \right) \quad (11)$$

Numerical results: The accuracy of the analytical results obtained above was corroborated with simulation results for a large range of values of the parameters η, κ, μ, α and m. Fig. 1 shows the results obtained for some selected values (similar trends were observed for other values).

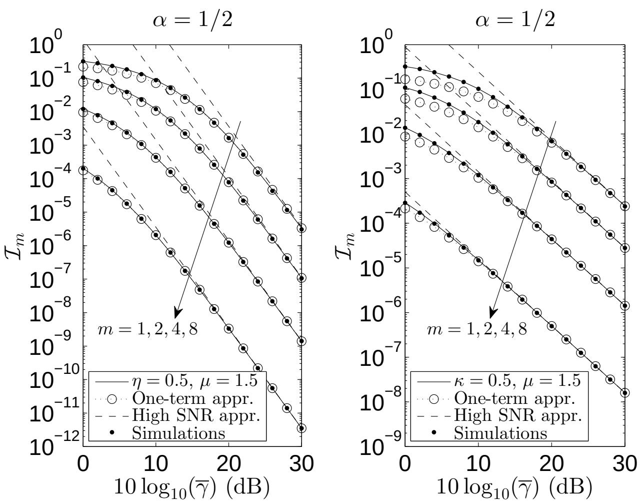

Fig. 1. Comparison of analytical and simulation results

As it can be appreciated, the obtained analytical results provide an excellent (virtually perfect) accuracy. While based on an approximation to the Gaussian Q-function, the expressions in (6) and (9), represented by solid lines, cannot be distinguished from the exact results obtained by simulation. This remarkable accuracy is due to the accuracy of the employed approximation along with the quick convergence of the infinite series in (6) and (9). As a matter of fact, the approximations obtained in (7) and (10) by truncating the infinite series to only their first term can actually provide a very close approximation to the exact results. Moreover, the asymptotic approximations in (8) and (11) provide not only a simple and tractable algebraic form but also an equally remarkable level of accuracy under high SNR conditions, which constitutes (as previously mentioned) the main SNR region of interest in the operation and design of wireless communication systems.

It is worth noting that the use of [6, eq. (3.462.1)] in order to obtain the presented analytical results implicitly assumes that 2μ (for η-μ) and μ (for κ-μ) take integer values. This assumption is valid in physical fading models since μ is related with the (integer) number of multipath clusters. However, 2μ and μ (for η-μ and κ-μ, respectively), can in practice take non-integer values for a number of reasons (the reader is referred to [1] for details). Interestingly, the analytical results presented in this Letter were observed to provide the same remarkable level of accuracy for both integer and non-integer values (i.e., virtually perfect results as shown in Fig. 1), which enables their use without any restrictions.

Given the excellent accuracy of the obtained analytical results along with the algebraic simplicity and tractability of their approximated forms, the expressions provided in this Letter constitute adequate tools for the average error performance evaluation over η-μ and κ-μ fading channels.

Conclusion: This Letter has contributed novel closed-form expressions for the average of arbitrary powers of the Gaussian Q-function under η-μ and κ-μ fading channels. The obtained expressions are based on approximations but in practice provide an excellent (virtually perfect) accuracy, comparable to the exact results. The presented expressions are useful in the evaluation of the average error performance of wireless communication systems over fading channels in a wide range of scenarios, including cases not covered by previous known results.

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